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Linear algebra – long test – solutions

This document contains problems from the long test along with several variants of their solutions. Moreover, I used red color to comment on these solutions. Some additional explanations (which you are not required to write every time) are also in red.

Problem 1. Let $V := \operatorname{lin}\big((3,2,3,4),(1,1,1,2),(5,3,6,3)\big)$.

(a) Find a basis and the dimension of V.

(b) Find a system of linear equations for which V is the set of solutions.

Solution I. (a) Let us use Gaussian elimination to simplify the collection of vectors spanning V. Since this is quite algorithmic, I omit the calculations here.

$$\begin{bmatrix} 3 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 5 & 3 & 6 & 3 \end{bmatrix} \rightsquigarrow \ldots \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

In the obtained matrix, rows are clearly linear independent. Since elementary operations on rows do not change the row span, we infer that the system $(1,0,0,3)$, $(0,1,0,2)$, $(0,0,1,-3)$ is a basis of V.

To justify linear independence, one should consider the equality

$$a(1,0,0,3)+b(0,1,0,2)+c(0,0,1,-3)=(0,0,0,0).$$

Since it is equivalent to

$$\begin{cases} a=0 \\ b=0 \\ c=0 \\ 3b+2b-3c=0 \end{cases}$$

it indeed implies $a=b=c=0$.

Since elementary operations on rows do not change linear dependence of rows, at the same time we can conclude that the vectors $(3,2,3,4)$, $(1,1,1,2)$, $(5,3,6,3)$ are also linearly independent and thus form a basis of V.

(b) Since V is the linear span of $(1,0,0,3)$, $(0,1,0,2)$, $(0,0,1,-3)$, it is by definition the space of vectors of the type $(x_1,x_2,x_3,x_4)=(a,b,c,3a+2b-3c)$. This can be described by just one equation

$$x_4 = 3x_1+2x_2-3x_3.$$

Solution II. (a) We shall check that the given vectors are linearly independent. To this end, we consider the equality

$$a(3,2,3,4)+b(1,1,1,2)+c(5,3,6,3)=(0,0,0,0).$$

Let us write it as a system of equations and solve it:

$$\begin{cases} 3a & +b & +5c = 0 \\ 2a & +b & +3c = 0 \\ 3a & +b & +6c = 0 \\ 4a & +2b & +3c = 0 \end{cases} \rightsquigarrow \begin{array}{ccc|c} 3 & 1 & 5 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 1 & 6 & 0 \\ 4 & 2 & 3 & 0 \end{array} \rightsquigarrow \dots \rightsquigarrow \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \rightsquigarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Luckily, the only solution is $a = b = c = 0$. This means that $(3, 2, 3, 4)$, $(1, 1, 1, 2)$, $(5, 3, 6, 3)$ are linearly independent and thus form a basis of V.

While computationally simpler, this method has two disadvantages.

First, if the obtained system has a nonzero solution (luckily, this was not the case above), then our vectors are linearly dependent and thus they are not a basis. On the other hand, we know *there always is* a basis, we just failed at finding one.

Second, even when our vectors are linearly independent, this method can only give us a proof that they are, but it does not give us any better basis. In particular, we do not obtain any description of V by linear equations.

Problem 2. Let $W \subseteq \mathbb{R}^4$ be the set of solutions of the system

$$\begin{cases} 3x_1 & +2x_2 & +3x_3 & +4x_4 & = 0 \\ x_1 & +x_2 & +x_3 & +2x_4 & = 0 \\ 5x_1 & +3x_2 & +6x_3 & +3x_4 & = 0 \end{cases}$$

(a) Find a general solution of this system..

(b) Find a basis and the dimension of W.

The system can be solved using Gaussian elimination. I skip the calculations once more. Do notice that they are exactly the same as in Problem 1.

$$\left.\begin{array}{cccc|c} 3 & 2 & 3 & 4 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 5 & 3 & 6 & 3 & 0 \end{array}\right. \rightsquigarrow \ldots \rightsquigarrow \left.\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right. \rightsquigarrow \begin{cases} x_1 = -3x_4 \\ x_2 = -2x_4 \\ x_3 = 3x_4 \end{cases}$$

There is exactly one free (independent) variable in our general solution (i.e., x_4), so W has dimension 1. To obtain its basis, we can take $x_4 = 1$ and then compute other coordinates according to our general solution. This gives us a basis consisting of just one vector $(-3, -2, 3, 1)$.

To justify that this one vector forms a basis of W, one can rewrite W in a different form:

$$\begin{aligned} W &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = -3x_4,\ x_2 = -2x_4,\ x_3 = 3x_4\} \\ &= \{(-3x_4, -2x_4, 3x_4, x_4) : x_4 \in \mathbb{R}\} \\ &= \{x_4(-3, -2, 3, 1) : x_4 \in \mathbb{R}\} \\ &= \operatorname{lin}\big((-3, -2, 3, 1)\big) \end{aligned}$$

Now the system consisting of one vector $(-3, -2, 3, 1)$ is linearly dependent (as it is nonzero) and it spans W, so it forms a basis of W.

Problem 3. Let $\alpha = (0, 3, 2, 0)$.

(a) Does α belong to V? If the answer is *yes*, give the coordinates of α in the basis found in Problem 1.

(b) Does α belong to W? If the answer is *yes*, give the coordinates of α in the basis found in Problem 2.

(b) Since W is given by a system of equations, it is trivial to check whether $\alpha \in W$. Setting $x_1 = 0$, $x_2 = 3$, $x_3 = 2$, $x_4 = 0$, we see that none of the three equations is satisfied, so α does not belong to W.

Actually, it would be enough to check that α does not satisfy one of the equations.

Keep in mind that if the system given in the problem were too complicated, we might just as well use the general solution instead (as it is an equivalent system).

(a) Once we have a description of V by a linear equation, the condition $\alpha \in V$ is also easy to check. Let us notice that the vector $\alpha = (0, 3, 2, 0)$ indeed satisfies the equation $x_4 = 3x_1 + 2x_2 - 3x_3$, and so it belongs to V.

To give its coefficient in the basis found before, we solve a system of equations:

$$a(1,0,0,3) + b(0,1,0,2) + c(0,0,1,-3) = (0,3,2,0)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 3 & 2 & -3 & 0 \end{array}\right) \rightsquigarrow \dots \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{cases} a = 0 \\ b = 3 \\ c = 2 \end{cases}$$

This means that α has coordinates $(0, 3, 2)$ in this basis.

Actually, since this system has a solution, we learn at the same time that α is a linear combination of our vectors (with coefficients $0, 3, 2$), and hence belongs to V. This makes checking the equation $x_4 = 3x_1 + 2x_2 - 3x_3$ redundant. However, it is usually good to check before starting any complicated calculations.

Alternative solution. (b) If the description of V by linear equations is not available, we can still check by definition whether α lies in the linear span of given vectors:

$$a(3,2,3,4) + b(1,1,1,2) + c(5,3,6,3) = (0,3,2,0)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 3 & 1 & 5 & 0 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 6 & 2 \\ 4 & 2 & 3 & 0 \end{array}\right) \rightsquigarrow \dots \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{cases} a = -7 \\ b = 11 \\ c = 2 \end{cases}$$

Since there is a solution, α belongs to V.

Moreover, if we have decided to choose $(3, 2, 3, 4)$, $(1, 1, 1, 2)$, $(5, 3, 6, 3)$ as our basis in Problem 1, the calculation above already gives us coordinates in this basis: $(-7, 11, 2)$.